

## ON THE PARALLEL SPECTRUM IN MHD TURBULENCE

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### ABSTRACT

Anisotropy of MHD turbulence has been studied extensively for many years, most prominently by measurements in the solar wind and high resolution simulations. The spectrum parallel to the local magnetic field was observed to be steeper than perpendicular spectrum, typically  $k^{-2}$ , consistent with the widely accepted Goldreich & Sridhar (1995) model. In this Letter I looked deeper into the nature of the relation between parallel and perpendicular spectra and argue that this  $k^{-2}$  scaling has the same origin as the  $\omega^{-2}$  scaling of Lagrangian frequency spectrum in strong hydrodynamic turbulence. This follows from the fact that Alfvén waves propagate along magnetic field lines. It is now became clear that the observed anisotropy can be argued without invocation of the “critical balance” argument and is more robust that was previously thought. The relation between parallel (Lagrangian) and perpendicular (Eulerian) spectra is inevitable consequence of strong turbulence of Alfvén waves, rather than a conjecture based on the uncertainty relation. I tested this using high-resolution simulations of MHD turbulence, in particular I verified that the cutoff of the parallel spectrum scales as Kolmogorov timescale, not lengthscale.

*Subject headings:* magnetohydrodynamics — turbulence

### 1. INTRODUCTION

Astrophysical and space plasmas are well-conductive, can be described as a magnetohydrodynamic fluid, which is usually turbulent (Armstrong et al. 1995; Biskamp 2003; Beresnyak & Lazarian 2014). At the same time high quality measurements in the solar wind has been available for more than two decades (Goldstein et al. 1995). The presence of large scale magnetic field is expected to change the dynamics dramatically. Analytic weak turbulence theory (Galtier et al. 2000) found that turbulent cascade proceeds in the direction perpendicular to the mean field, resulting in stronger turbulence, which should eventually break down weak turbulence on sufficiently small scales. Similar qualitative arguments earlier lead Goldreich & Sridhar (1995) to suggest that the inertial range of MHD turbulence will be strong turbulence. They also argued that such turbulence will be “critically balanced”, or marginally strong, with the linear propagation term always contributing comparably with the nonlinear interaction term predicting the  $k_{\parallel} \sim k_{\perp}^{2/3}$  anisotropic cascade, which has found support in numerics, e.g., Cho & Vishniac (2000); Maron & Goldreich (2001). The  $k_{\perp}^{-5/3}$  perpendicular spectrum and  $k_{\parallel} \sim k_{\perp}^{2/3}$  anisotropy results is a  $k_{\parallel}^{-2}$  parallel spectrum. One observation is paramount for our understanding of this parallel spectrum. While Goldreich & Sridhar (1995) suggested a closure model predicting the  $k_{\parallel} \sim k_{\perp}^{2/3}$  anisotropy in the frame associated with the global mean field, it was not observed in Cho & Vishniac (2000), rather this anisotropy was observed in the structure function measurement performed in the frame associated with the local magnetic field. Similarly, Horbury et al. (2008) observed the  $k_{\parallel}^{-2}$  parallel scaling using the wavelet technique and associating parallel direction to the direction of the local field.

Given the importance of the parallel spectrum for a variety of phenomena, e.g. resonant scattering of solar energetic particles, the measurement of the parallel spectrum in the solar wind attracted considerable attention, see, e.g., Horbury et al. (2008); Podesta (2009); Osman & Horbury (2009); Wicks et al. (2010); Luo & Wu (2010). These measurements followed the prescription of the local field direction and generally confirmed the  $k_{\parallel}^{-2}$  scaling, however, the debate surrounding the critical balance argument and the nature of anisotropy continues Grappin & Müller (2010); Grappin et al. (2013).

In this Letter I will argue that there is a conceptually simpler way to look at the MHD anisotropy, namely as a relation between Lagrangian and Eulerian spectra. I will also introduce the statistically averaged one-dimensional spectrum along the field line and show that high resolution numerics support steep parallel spectra, consistent with  $k_{\parallel}^{-2}$ , just like in the solar wind measurements.

### 2. STRONG TURBULENCE AND LAGRANGIAN SPECTRUM

Strong turbulence was suggested to be scale-local and self-similar in Kolmogorov (1941), which lead to his  $k^{-5/3}$  Eulerian power spectrum of velocity perturbations. Another basic spectrum of hydrodynamic turbulence is called Lagrangian frequency spectrum, which statistically evaluates how the velocity of the fluid element changes with time. Assuming that the dot product of the total time derivative of the velocity and the velocity vector itself is a work done upon a fluid element, one could estimate  $\delta \mathbf{v}_{\tau} \cdot \delta \mathbf{v}_{\tau} / \tau$  as the turbulence energy cascade rate per unit mass  $\epsilon$ , measured in  $\text{cm}^2/\text{s}^3$ , also known as dissipation rate. More precisely, in stationary turbulence the second-order structure function of veloc-

ity should satisfy:

$$SF(\tau) = \langle (\mathbf{v}(t + \tau) - \mathbf{v}(t))^2 \rangle \approx \epsilon \tau \quad (1)$$

in the inertial range, where  $\mathbf{v}(t)$  is a velocity of a given fluid element. Such time structure function is dual to the frequency spectrum of  $E(\omega) = \epsilon \omega^{-2}$  (Landau & Lifshitz 1944; Corrsin 1963; Tennekes & Lumley 1972). The cut-off of this spectrum is associated with the timescale of critically viscously damped eddies, the Kolmogorov timescale  $\tau_\eta = (\nu/\epsilon)^{1/2}$ , which has a different dependence on the Reynolds number  $\text{Re} = v_L/\nu$  compared to Kolmogorov length scale  $\eta = (\nu^3/\epsilon)^{1/4}$ , the cutoff of the Eulerian spectrum.

Magnetized dynamics is qualitatively different from hydrodynamics in that locally there is always a propagating wave characteristic. In particular, following a fluid element, we may find oscillations associated with the wave-train that propagates through this fluid element in the direction of the local mean magnetic field, which makes classic Lagrangian measurement of limited value. The Lagrangian evolution in MHD takes on a different meaning, therefore. The Alfvén perturbations can be decomposed into Elsässer components  $\mathbf{w}^\pm = \mathbf{v} \pm \mathbf{B}/\sqrt{4\pi\rho}$ , each of which propagate either along or against the local field direction, i.e. along the magnetic field line. The functional dependence of such perturbations will take the form  $f(s \mp v_A t)$  in the absence of interaction, where  $s$  is a distance along the field line. If the nonlinear interaction is present, the trajectory  $s = \pm v_A t$  would act as an analogy to hydrodynamic fluid element trajectory if we want to study Lagrangian dynamics in MHD.

The above argument suggests that following evolution of  $\mathbf{w}^+$  and  $\mathbf{w}^-$  along the field line in fixed time and in the direction positive in  $s$  would be equivalent to following evolution of  $\mathbf{w}^+$  backward in time and  $\mathbf{w}^-$  forward in time. This simple argument has been already fruitful in explaining the asymmetric Richardson diffusion of magnetic field lines (Beresnyak 2013). As far as frequency spectrum goes, the sign of time is unimportant and any measurement of power spectrum along the field line of either  $\mathbf{v}$ ,  $\mathbf{B}$  or  $\mathbf{w}^\pm$  will be analogous to Lagrangian frequency spectrum with frequency  $f$  replaced by the wavenumber  $f/v_A$ :  $E(k_\parallel) \sim \epsilon v_A^{-1} k_\parallel^{-2}$ . The spatial structure function will be expressed correspondingly as  $SF_\parallel(l) = \epsilon l v_A^{-1}$ .

Another way to argue the  $k_\parallel^{-2}$  parallel scaling is the dimensional argument using the Alfvén symmetry of reduced MHD used in Beresnyak (2012b). Indeed, this symmetry dictates that changing  $v_A$  while keeping  $k_\parallel v_A$  constant leave equations unchanged. Therefore, one must keep energy  $E(k_\parallel) dk_\parallel$  constant under such transformation, which require that  $E(k_\parallel) \sim v_A^{-1}$ . Using scale-locality, i.e., assuming that the spectrum can only depend on  $v_A$ ,  $\epsilon$  and  $k_\parallel$ , I arrive at

$$E(k_\parallel) = C_\parallel \epsilon v_A^{-1} k_\parallel^{-2}, \quad (2)$$

where  $C_\parallel$  is dimensionless constant. Logically, this dimensional argument is a restatement of the Lagrangian spectrum argument. Note that the parallel second-order spectrum scales linearly with the dissipation rate  $\epsilon$ , similarly to the *third-order* Eulerian scaling and not to  $\epsilon^{2/3}$

TABLE 1  
THREE-DIMENSIONAL MHD AND RMHD SIMULATIONS

Run	$N^3$	Dissipation	$v_A$	$\epsilon$	$\eta$	$v_A \tau_\eta$
MHD1	1536 <sup>3</sup>	$-5 \cdot 10^{-10} k^4$	0.73	0.091	0.0021	0.026
MHD2	1536 <sup>3</sup>	$-6.2 \cdot 10^{-10} k^4$	1.53	0.728	0.0018	0.025
M1	1024 <sup>3</sup>	$-1.75 \cdot 10^{-4} k^2$	1	0.06	0.0031	0.044
M2	2048 <sup>3</sup>	$-7 \cdot 10^{-5} k^2$	1	0.06	0.00155	0.028
M3	4096 <sup>3</sup>	$-2.78 \cdot 10^{-5} k^2$	1	0.06	0.00077	0.017
M1H	1024 <sup>3</sup>	$-1.6 \cdot 10^{-9} k^4$	1	0.06	0.0030	0.045
M2H	2048 <sup>3</sup>	$-1.6 \cdot 10^{-10} k^4$	1	0.06	0.00152	0.029
M3H	4096 <sup>3</sup>	$-1.6 \cdot 10^{-11} k^4$	1	0.06	0.00076	0.018

scaling of the second-order Eulerian spectrum.

Unlike Reduced MHD, full MHD have no exact Alfvén symmetry. The arguments in favor of using it in the inertial range are still quite compelling (Beresnyak 2012b). It is interesting to check if the parallel spectrum still follow Eq. (2) not only in Alfvénic MHD, but in the general MHD case. Especially interesting is the case with zero mean magnetic field where the  $v_A$  will be determined only by local fluctuations.

### 3. NUMERICS

First half of the numerical data is from my DNS of strong reduced MHD turbulence (Beresnyak 2014), which are well-resolved statistically stationary driven simulations intended to precisely calculate averaged quantities. Note that reduced MHD, i.e. Alfvén dynamics, does not depend on plasma pressure and can be applied to situations with different values of plasma  $\beta$ , from zero to infinity. I list the most important parameters of these simulations in code units in Table 1 under rows M1-3 and M1H-3H. The only difference between M1-3 and M1H-3H was that the latter were performed with higher order diffusivities. Additionally, I performed simulations of statistically isotropic driven incompressible MHD turbulence with zero mean field with parameters presented in Table 1 rows MHD1-2. For all cases I have calculated the spectra along the magnetic field line, and for the reduced MHD cases I additionally have calculated the one-dimensional spectra along the  $x$  direction which was the global mean field direction.

Three dimensional numerics have modest  $\text{Re}$  and are always affected by the finite  $\text{Re}$  effects. I used a rigorous scaling study method, fairly common in the analysis of experimental data and DNS (Sreenivasan 1995; Gotoh et al. 2002; Kaneda et al. 2003; Beresnyak 2012a, 2014), which compares spectra from simulations with several different  $\text{Re}$  values on the same plot with dimensionless axes. The parallel spectrum was plotted vs dimensionless wavenumber  $k v_A \tau_\eta$  and compensated by  $k^2 \epsilon^{-1} v_A$  to see how the scaling is consistent with (2). This measurement is presented on Fig 1. For the reduced MHD case the spectra collapsed on the dissipation scale, corresponding to an overall scaling of  $k^{-2}$ .

Given that reduced MHD has precise Alfvén symmetry and the requirement of turbulence to be strong on the outer scale assumes a certain value of  $\epsilon$ , it does not allow us to check the linear scaling with  $\epsilon$  in Eq. (2), as I couldn't vary  $\epsilon$  in M1-3. I used statistically isotropic MHD simulations with zero mean field MHD1-2, for which Alfvén symmetry is absent and the inertial range

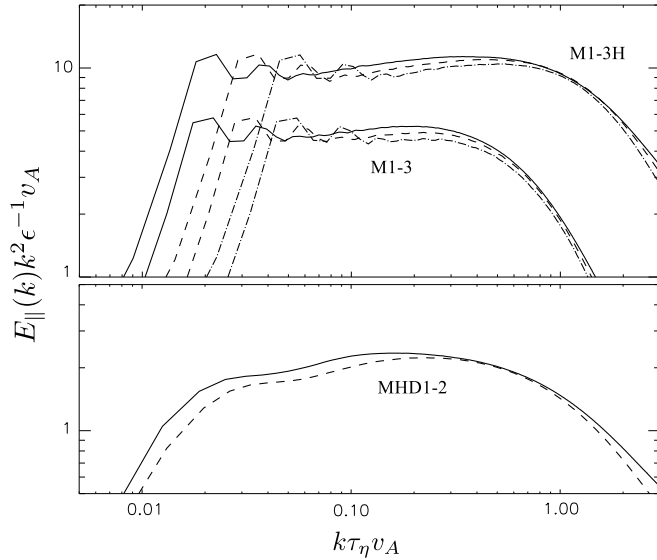


FIG. 1.— Energy spectrum along the magnetic field line compensated by the theoretical scaling  $\epsilon k_{\parallel}^{-2}$  (2). Solid, dashed and dash-dotted are spectra from 4096<sup>3</sup>, 2048<sup>3</sup> and 1024<sup>3</sup> simulation correspondingly on the upper plot. The M1-3H has been multiplied by a factor of two to separate the curves. On the lower plot dashed and solid are MHD1 and MHD2 correspondingly.

scaling (2) can not be rigorously argued based on units. Despite that, the standard argumentation, introduced by Iroshnikov (1964); Kraichnan (1965) is that the RMS magnetic field can play the role of the local mean field and this could still be regarded as the *strong mean field* limit. I will conjecture that the parallel spectrum will still follow Eq. (2) in the inertial range in this case as well. In the MHD case I used simulations with different  $\epsilon$  and substituted the RMS field instead of  $v_A$  in Eq. (2). Fig. 1 demonstrates that there's an inertial range convergence to  $k^{-2}$  even in this zero mean field case. The linear scaling with  $\epsilon$ , not  $\epsilon^{2/3}$  is also confirmed.

Another possible spectral measurement is with respect to the global mean field. We do not expect such scalings to deviate significantly from the perpendicular scalings for the following reason: Alfvén waves propagate along the local field direction which deviates by an angle of  $\delta B_L/B_0$  from  $\mathbf{B}_0$ , while the angular anisotropy in this frame is  $\delta B_L/B_0$ , with inertial range values of  $\delta B_L$  much smaller than the outer scale value of  $\delta B_L$ . It follows that the anisotropy will be washed out. Fig. 2 presents a measurement of spectrum along the  $x$  – global mean field direction. It is grossly consistent with  $-5/3$ , i.e. the perpendicular spectral scaling observed in Beresnyak (2014).

#### 4. DISCUSSION

Critical balance refers to the interaction parameter  $\xi = \delta v \lambda_{\parallel}/v_A \lambda_{\perp}$  being around unity in strong MHD turbulence. It has been first argued based on the uncertainty relation between the wave frequency and the cascade timescale in Goldreich & Sridhar (1995) and had been restated in various forms, including the decorrelation argument by Gruzinov (Maron & Goldreich 2001). While the plausibility arguments like this are certainly useful in qualitative understanding, their apparent gen-

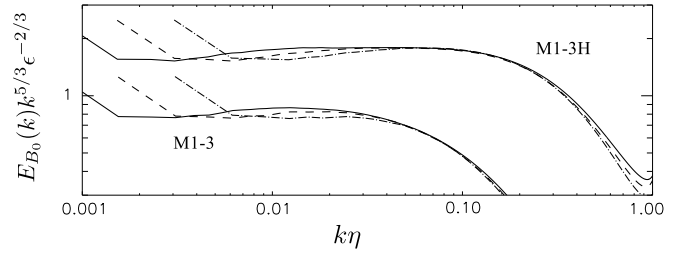


FIG. 2.— The spectra along the global mean field in M1-3, M1-3H. The M1-3H spectra has been multiplied by a factor of two. This plot demonstrates that this energy spectrum scaling is consistent with  $-5/3$ , i.e. the same as the perpendicular scaling.

erality is problematic. E.g. the decorrelation argument does not explicitly refer to nonlinear interaction, however, it could not be generally valid, as pure propagating solutions with  $\xi \gg 1$ , strong Alfvénic waves, do exist. The naive application of the uncertainty relation argument fails, e.g., in imbalanced turbulence, where it predicts that the anisotropy of the stronger Elsässer component should be higher than the anisotropy of the weaker component, while in reality the opposite is true (Beresnyak & Lazarian 2008, 2009b). The new argument, presented in this Letter, circumvents this problem by noticing that the energy cascade is manifested both in space and time domains, also the parallel direction is equivalent to the time domain. Therefore the well-known anisotropy relation  $k_{\parallel} \sim k_{\perp}^{2/3}$  is simply the correspondence between space domain (Eulerian) and frequency domain (Lagrangian) spectra. The old arguments required that the average  $\xi$  must be close to unity, while the new argument only requires that the average  $\xi$  is a dimensionless, scale-independent quantity, i.e. a constant, similar to Kolmogorov constant.

Most observational data from the solar wind has been pointing to the  $k^{-2}$  parallel spectrum. For example Horbury et al. (2008) used a wavelet technique to follow the local field direction and obtained  $k^{-2}$ . This has been further improved in Wicks et al. (2010) and compared with the global Fourier spectra. Podesta (2009) obtained similar results with wavelets and demonstrated scale-dependent anisotropy. The structure function measurement in Luo & Wu (2010) again confirmed the same scaling. Multi-spacecraft measurements allowing better coverage of  $k$ -space (Osman & Horbury 2009) also confirmed  $k^{-2}$ . Earlier measurements in the *global* frame, e.g. Matthaeus et al. (1990) reported *scale-independent* anisotropy, which, as I argued above, is consistent with theory and numerics as well. As far as numerics go, the measurements along the local field direction gave the  $k^{-2}$  slope, see, e.g., Cho & Vishniac (2000); Maron & Goldreich (2001); Beresnyak & Lazarian (2009b,a); Chen et al. (2011); Beresnyak (2012a), while the measurements in the global frame gave scale-independent anisotropy, see, e.g. Grappin & Müller (2010) or my Fig 2. The robustness of the critical balance with properly defined nonlinear timescale has been recently discussed in Mallet et al. (2014).

Recently, the debate on the parallel spectrum has been revived, in particular Turner et al. (2012) measured quasi-isotropic spectrum in the solar wind after

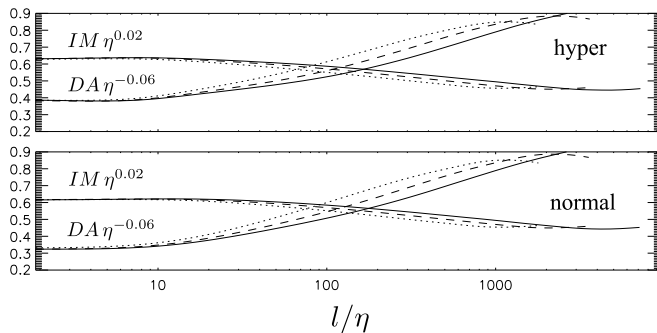


FIG. 3.— Scaling study of alignment measures  $DA = \langle |\delta \mathbf{v} \times \delta \mathbf{b}| \rangle / \langle |\delta v \delta b| \rangle$  and  $IM = \langle |\delta(w^+)^2 - \delta(w^-)^2| \rangle / (\delta(w^+)^2 + \delta(w^-)^2)$  from M1-3H (top) and M1-3 (bottom). The alignment slopes converge to relatively small values, e.g., 0.06 for  $DA$  which is smaller than 0.25, predicted by Boldyrev (2006). See also Beresnyak & Lazarian (2009a); Beresnyak (2011, 2012a).

filtering out discontinuities, while Grappin et al. (2012, 2013) suggested a new model with “ricochet” cascade that effectively fills parallel direction and results in the same slope, as in perpendicular direction, citing Grappin & Müller (2010); Turner et al. (2012) as motivation. My numerical data strongly disfavors this model, as the observed  $-2.0 \div -1.9$  parallel spectral slope is much steeper than either  $-5/3$  or the  $-1.5$  suggested in Grappin et al. (2012, 2013). It is also not clear why global measurement (Grappin & Müller 2010) should support the alternative model or whether filtering in Turner et al. (2012) interfere with the local field direction enough to destroy the weaker  $k^{-2}$  parallel spectrum.

Measurements of Lagrangian frequency spectrum in hydrodynamics has been performed by many authors, see, e.g., Yeung et al. (2006) and references therein, and showed correspondence with the theoretical  $\omega^{-2}$ . First direct measurement of Lagrangian frequency spectrum in statistically isotropic MHD turbulence has been performed in Busse et al. (2010) and tentatively confirmed the  $\omega^{-2}$  scaling, but the results from simulations with strong mean field were less clear. The connection between Lagrangian frequency spectrum and parallel spatial spectrum has not been made in Busse et al. (2010), also as I argued above, the classic Lagrangian measurement could be meaningless in MHD as high-frequency perturbations cross the fluid element causing oscillatory velocity changes not associated with the energy cascade. In this Letter I argued that the measurement

of the spectrum *along the magnetic field line* is similar to the measurement of Lagrangian frequency spectrum and therefore the Goldreich & Sridhar (1995) scale-dependent anisotropy is a simple relation between Eulerian and Lagrangian spectra.

Using scaling study argument I found that the best convergence correspond to  $k^{-2}$ , however the deviation around 0.1 in scaling exponent on medium scales is evident on Fig. 1 and is somewhat interesting from theoretical viewpoint as a long-range finite-Re effect. It is clear that  $-1.9$  slope on the medium scales on Fig. 1 is still much steeper than Eulerian  $-5/3 \approx -1.7$ , but what is the nature of this deviation? Firstly, the prediction of the models with so-called dynamic alignment modifies only perpendicular spectrum, leaving parallel spectrum unchanged, at the expense of higher anisotropy (Boldyrev 2005, 2006). Secondly, even if such modification was suggested by some inertial-range theory, it would be inconsistent with my numerics, as the 0.1 correction is not universal and disappear with higher Re measurements.

In the past several years various spectral scalings, deviations from theoretically predicted scalings and alignment measures has been studied in some detail (Beresnyak & Lazarian 2009a; Beresnyak 2011, 2012b). The overall picture seems to be that while moderate Re shows scale-dependency of several alignment measures, normally in the range of 0.1 – 0.2, in the higher Re measurements these alignment measures flatten out and their slopes are fairly close to zero, see, e.g., Fig. 3. Similarly, the deviation from the expected perpendicular  $-1.7$  slope is around  $\sim 0.2$  in the medium scales, but disappeared when higher resolution data became available. The  $\sim 0.1$  deviation of the parallel slope fits nicely into this tendency. We see that the deviations from theoretical scalings had been observed so far only within around an order of magnitude in scale from the driving scale and modifications of theory of the inertial range, such as Chandran et al. (2014) are probably excessive at this point. Further solar wind measurements with better statistics and/or larger scale numerics will help to shed light on this problem.

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